Imposing a Lower Bound on Voter Turnout

Yoichi Hizen *
Masafumi Shinmyo **

HOPS Discussion Paper Series No.6
April 2007

* Faculty of Public Policy, Hokkaido University
  Kita-9 Nishi-7, Kita-ku, Sapporo, Hokkaido, 060-0809, JAPAN
  e-mail: hizen@econ.hokudai.ac.jp

** The Graduate School of Public Policy, Hokkaido University
  Kita-9 Nishi-7, Kita-ku, Sapporo, Hokkaido, 060-0809, JAPAN
Imposing a Lower Bound on Voter Turnout*

Yoichi Hizen† Masafumi Shinmyo‡
Hokkaido University Hokkaido University

First Draft: March 2005; This Draft: April 2007

Abstract

This paper analyzes a referendum whose outcome is valid only if the voter turnout ratio is greater than a predetermined value. Our game-theoretic model shows that this lower bound can induce strategic voters to abstain intending to spoil the outcome while avoiding losing the referendum. Therefore, its outcome may not reflect voters’ preferences accurately. As a referendum whose outcome was made invalid due to low turnout, a Japanese city Ishikari’s referendum on the consolidation with other municipalities is discussed.

1 Introduction

Voting is a way of eliciting the aggregate opinion from people who are given the voting rights. However, voters do not necessarily come to the poll. When the voter turnout is low, can we regard the outcome as reflecting voters’ opinion accurately? Probably the answer depends on what group of people abstain. That is, the vote distribution is a fair sample of the opinion of the whole population if voters abstain randomly, but it is not if particular people abstain.

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*We are grateful to Masaki Aoyagi, Yoshio Kamijo, Kazuharu Kiyono, Yosuke Oyama, Ken’ichi Shiozawa, Shingo Takagi and seminar participants at Osaka University and Waseda University for their useful comments. This research is partially supported by the Japanese Ministry of Education, Culture, Sports, Science and Technology, Grant-in-Aid for Young Scientists (B), 18730206, 2006. We are solely responsible for any remaining errors.

†Address: Faculty of Public Policy, Hokkaido University, North 9, West 7, Kita-ku, Sapporo, 060-0809 JAPAN [Email: hizen@econ.hokudai.ac.jp].

‡Address: Graduate School of Public Policy, Hokkaido University, North 9, West 7, Kita-ku, Sapporo, 060-0809 JAPAN.
Recently, Japanese policy makers seem more skeptical about voting outcomes yielded with low voter turnout. In their referendums which asked their citizens whether to approve or disapprove the consolidation of their municipalities with their neighboring ones, many city, town and village assemblies imposed a requirement on the voter turnout ratio which must be satisfied for the validity of the referendums: the outcome is valid only if the voter turnout ratio is greater than a predetermined value, which ranges from $33.3\%$ to $70\%$.\(^1\) As of January 14, 2005, seven referendums among them did not open their ballot boxes because their voter turnout ratios did not satisfy their validity conditions (Hokkaido Shinbun, January 17, 2005), and two more city and town, Ishikari (January 16, 2005) and Noichi (March 27, 2005), experienced the same result.

It is well recognized in the voting literature that voting rules affect voters’ behaviors and hence the voting outcomes: the most known is Duverger’s (1954) law and hypothesis. Does imposing a lower bound on voter turnout also affect voters’ behaviors? This paper tackles this question by constructing a Bayesian-Nash equilibrium model of voting decisions. We consider an yes-no referendum with a lower bound, and obtain a result that every voter votes sincerely in equilibrium if the lower bound is not imposed or if it is negligibly small, but various types of abstention happen if it is effectively high. Strategic abstention comes from the incentive for voters whose favorite alternative is less likely to win to spoil the outcome by decreasing the voter turnout ratio through abstaining, rather than helping the voter turnout ratio rise beyond the lower bound by going to the poll. Therefore, the outcome may not reflect voters’ preferences accurately. Our result is consistent with the impossibility theorem derived axiomatically by Cörte-Real and Pereira (2004). They show that under the validity condition, no voting rule can ensure representation, if abstention is possible, for a broad range of preference domain of abstainers.\(^2\) To discuss our theoretical implication, we also describe a Japanese city Ishikari’s referendum in detail as a recent example of referendum whose outcome was made invalid due to low voter turnout.

Why people vote is a traditional research question in the voting literature.\(^3\) The main argument is that the cost of going to the poll should exceed the expected benefit from affecting the outcome. Then do people abstain

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\(^1\)Imposing a lower bound on voter turnout is not limited to referendums regarding the consolidation of municipalities. Municipal bylaws of not a small number of cities, like Fujimi (Saitama Prefecture), Hiroshima, Otake (Hiroshima), Kiryu (Gunma) and Takahama (Aichi), impose it for any type of referendum.

\(^2\)Cörte-Real and Pereira (2004) also describe the rules of referendums employed by European countries.

\(^3\)See Dhillon and Peralta (2002) for a survey of theories of voter turnout.
even without the cost of going to the poll? Feddersen and Pesendorfer (1996, 1999) show that the lack of information about candidates induces uninformed voters to abstain with the intention of delegating the appropriate choice to informed voters. We provide another cause of strategic abstention, which is the voting rule itself. Our model-building strategy is classified in the stream of Bayesian models of voting decisions initiated by Ledyard (1981). Among them, the most closely related one to ours is Palfrey’s (1989). He shows that Duverger’s law, which states that only two candidates are serious in single-seat elections, holds in a three-candidate election when the number of voters converges to infinity. We do not deal with asymptotic properties, but simplify his model so as to introduce the lower bound.

This paper is organized as follows. Section 2 builds the model. Section 3 solves it to obtain propositions on the voting outcomes. Section 4 provides a numerical example. Section 5 describes Ishikari’s referendum in detail to discuss the implications of the model. Section 6 concludes.

2 The Model

2.1 Basic Structure

We consider an yes-no referendum. A positive number of voters \(m > 0\) are independently assigned preferences regarding the subject of the referendum: each of them is assigned ”yes” (abbreviated as ”y”) with probability \(s \in (0, 1)\) and ”no” (abbreviated as ”n”) with \(1 - s\). The number of voters \(m\) and the probability \(s\) are common knowledge among voters, but each voter does not know the realizations of other voters’ preferences. Each voter has one vote. After assigned their own preferences only, voters simultaneously vote for yes or no or abstain. That is, each voter’s pure strategy is a function from her own preference to her voting behavior \(\{y, n\} \rightarrow \{y, n, a\}\), where ”a” represents abstention. Mixed strategies are also allowed.

Let \(m_i (i = y, n, a)\) denote the number of voters who have chosen \(i\) in the ex-post sense. The outcome of the referendum is valid only if the voter turnout is greater than or equal to a predetermined number. In other words, the outcome is valid if \(m_y + m_n \geq rm\) while invalid otherwise, where \(r \in [0, 1]\) is the lower bound of the voter turnout ratio. For the ease of calculation of pivot probabilities, we impose the following two assumptions on the values of \(m\) and \(r\):

**A1:** \(m\) is an odd number.

**A2:** \([rm]\) is an odd number, where \([x]\) represents the smallest integer greater than or equal to \(x\).
When the outcome is valid, yes wins if \( m_y > m_n \) while no wins if \( m_y < m_n \). Any tie is broken randomly.

Each voter enjoys a benefit \( v_{\text{win}} \) if the voting outcome is in accordance with her preference while \( v_{\text{lose}} \) if not. We normalize benefits by assuming \( v_{\text{win}} = 1 \) and \( v_{\text{lose}} = 0 \). If the outcome is invalid, each voter gets \( v \in (0,1) \). That is, the invalid outcome is worse than her favorite outcome but better than the opposite outcome. Going to the poll costs nothing. Each voter is instrumental: she only cares about how her vote affects the outcome and chooses her strategy in order to maximize her expected benefit.

If every voter is choosing a strategy which maximizes her expected benefit given other voters’ strategies, then the strategy profile is called Bayesian Nash equilibrium. A Bayesian Nash equilibrium is called symmetric if any two voters with the same preference choose the same behavior. In the analysis, we focus on symmetric Bayesian Nash equilibria.

Voting for the opposite alternative is a weakly dominated strategy for any voter: voting for her favorite alternative has the same effect as voting for the opposite one in increasing the voter turnout ratio, but the former can change the winner from the opposite one to her favorite one while the latter has the opposite effect in this point. Therefore, our analysis also concentrates on the equilibria in which voters either vote for their favorite alternative or abstain.\(^4\)

### 2.2 Pivot Probabilities

Let us describe the pivot probabilities for each vote. Let \( \pi_i \ (i = y, n) \) denote the probability that a type-\( i \) voter chooses action \( i \). Then \( 1 - \pi_i \) is the probability of abstention. A vote for \( i \) can affect the outcome in the following three ways. The first is to make the outcome valid with \( i \)'s win. This happens with certainty if, except one vote, \( m_y + m_n = \lceil rm \rceil - 1 \) and \( m_i \geq m_j \ (j \neq i, j = y, n) \) hold, and with probability \( 1/2 \) if, except one vote, \( m_y + m_n = \lceil rm \rceil - 1 \) and \( m_i = m_j - 1 \) hold. However, the latter event never occurs under A2. Hence, this probability is written as

\[
p_i = \sum_{k=0}^{\lceil rm \rceil - 1} \frac{(m - 1)!}{k!(\lceil rm \rceil - 1 - k)!(m - \lceil rm \rceil)!} \sigma_i^{\lceil rm \rceil - 1 - k} \sigma_j^k \sigma_a^{m - \lceil rm \rceil}
\]

where \( \sigma_y = s \pi_y, \sigma_n = (1 - s) \pi_n \) and \( \sigma_a = 1 - \sigma_y - \sigma_n \).

The second is to make the outcome valid with \( j \)'s win. This happens with certainty if, except one vote, \( m_y + m_n = \lceil rm \rceil - 1 \) and \( m_j \geq m_i + 2 \)

\(^4\)See Palfrey (1989) for a discussion about eliminating weakly dominated strategies in this type of voting model.
hold, and with probability 1/2 if, except one vote, \( m_y + m_n = \lfloor rm \rfloor - 1 \) and \( m_j = m_i + 1 \) hold. The latter event never occurs under A2. Hence, this probability is written as

\[
q_i = \frac{\lfloor rm \rfloor - 1}{2} \sum_{k=0}^{\lfloor rm \rfloor - 1} \frac{(m-1)!}{k!([rm] - 1 - k)!(m - [rm])!} \sigma^k_j \sigma_j^{\lfloor rm \rfloor - 1 - k} \sigma^m_a^{m - [rm]}.
\]

The last is for a vote for \( i \) to change the winner from \( j \) to \( i \) when the outcome is valid even without that vote. This happens with probability 1/2 if, except one vote, \( m_j + m_n \geq \lfloor rm \rfloor \) and either \( m_i = m_j - 1 \) or \( m_y = m_n \) hold. By A1, this probability is written as

\[
t_i = \frac{1}{2} \left( \sum_{k=\lfloor rm \rfloor - 1}^{\lfloor rm \rfloor - 2} \frac{(m-1)!}{k!(k+1)!(m - 2k - 1)!} \sigma^k_j \sigma_{j+1}^{\lfloor rm \rfloor - 1 - k} \sigma^m_a^{m - 2k - 2} \right) + \frac{1}{2} \sum_{k=\lfloor rm \rfloor - 1}^{\lfloor rm \rfloor - 1} \frac{(m-1)!}{k!k!(m - 2k - 1)!} \sigma^k_j \sigma_{j+1}^{\lfloor rm \rfloor - 1 - k} \sigma^m_a^{m - 2k - 1}.
\]

Given these probabilities, type-\( i \) voters vote for \( i \) only if

\[
(1 - v)p_i + t_i \geq vq_i.
\]

## 3 Analysis

In this section, we derive symmetric Bayesian Nash equilibria. We first deal with a benchmark where the lower bound is not imposed or ineffectively small. Then we examine how the lower bound affects voting behaviors if it is sufficiently large.

### 3.1 Ineffective Lower Bounds

If nothing is imposed on voter turnout (i.e., \( r = 0 \)), the outcome is always valid. Then voters care about the only pivot probability that their votes change the winner from one alternative to the other (i.e., \( p_i = q_i = 0 \) and \( t_i > 0, i = y, n \)). This is a well-known two-candidate election in which sincere voting is optimal even for strategic voters.

The only difference between \( r = 0 \) and \( r \in (0, 1/m] \) is that when nobody votes (i.e., \( m_y + m_n = 0 \)), one of yes and no wins under \( r = 0 \) while the outcome is invalid under \( r \in (0, 1/m] \). For this small value of \( r \in (0, 1/m] \), one vote is enough to make the outcome valid. In other words, a vote changes the
outcome from invalid to valid only when all other voters abstain. Therefore, a vote for $i$ is never accompanied by $j$’s ($j \neq i, j = y, n$) win when it makes the outcome valid (i.e., $q_i = 0$). So voters vote sincerely. We obtain our first proposition.

**Proposition 1.** For $r \leq 1/m$, the unique Bayesian Nash equilibrium is $(y \rightarrow y, n \rightarrow n)$.

As a result, the outcome, or the distribution of votes for *yes* and *no*, reflects voters’ preferences exactly.

### 3.2 Effective Lower Bounds

Next let us consider effective lower bounds. We can divide the range of effective lower bounds into two intervals, $r \in (1/m, (m - 1)/m]$ and $r \in ((m - 1)/m, 1]$. We first deal with the first interval, which is the main part of this paper. Then the second interval is examined, in which full voter turnout is required for the validity of the outcome.

#### 3.2.1 Case $r \in (1/m, (m - 1)/m]$

For this range of lower bound, one vote is not enough to make the outcome valid, nor is full turnout required. Therefore, regardless of one voter’s behavior, the outcome remains invalid if all other voters abstain, while it remains valid if all other voters vote. These facts imply that both zero turnout (i.e., $(y \rightarrow a, n \rightarrow a)$) and sincere voting (i.e., $(y \rightarrow y, n \rightarrow n)$) are realized in equilibrium.

Suppose that *yes*-voters vote sincerely while *no*-voters abstain (i.e., $(y \rightarrow y, n \rightarrow a)$). Under this strategy profile, changing the outcome from invalid to valid must be accompanied by *yes*’s win (i.e., $p_y, q_n > 0$ and $p_n = q_y = 0$). In addition, any valid outcome necessarily implies *yes*’s win (i.e., $t_i = 0, i = y, n$). Therefore, *yes*-voters vote sincerely without worrying about their votes resulting in *no*’s win while what each *no*-voter can do is to spoil the outcome by abstaining. That is, this strategy profile constitutes an equilibrium. Similarly, $(y \rightarrow a, n \rightarrow n)$ is also an equilibrium. The same logic suggests that neither $(y \rightarrow a, n \rightarrow n \& a)$ nor $(y \rightarrow y \& a, n \rightarrow a)$ constitutes an equilibrium because the group members using a mixed strategy will deviate to voting for their favorite alternative with certainty.

Now let us consider the remaining strategy profiles. Suppose $(y \rightarrow y, n \rightarrow n \& a)$. This strategy profile is incentive compatible if there exists $\pi_n \in (0, 1)$ which satisfies equation (1) with either equality or inequality for $i = y$ and
with equality for \( i = n \). These two conditions are combined into

\[
\frac{p_y + t_y}{p_y + q_y} \geq \frac{p_n + t_n}{p_n + q_n} = v. \tag{2}
\]

The equality (i.e., no-voters’ incentive constraint) determines the value of \( \pi_n \) as a function of four parameters, \( m, s, v \) and \( r \). For such a value of \( \pi_n \) to constitute an equilibrium, it must be between 0 and 1, and it must also satisfy the inequality (i.e., yes-voters’ incentive constraint). As \( \pi_n \) converges to 0, \( p_y \) and \( q_n \) converge to a positive value (i.e., \( \frac{(m-1)!}{[r_m-1]!(m-[r_m])!} s^{r_m-1}(1-s)^{m-[r_m]} \)) while other pivot probabilities to 0, which implies that \( y \)'s fraction converges to 1 while \( n \)'s fraction to 0. Since pivot probabilities are continuous in \( \pi_n \), therefore, at least for sufficiently small values of \( v \), we can find \( \pi_n \in (0,1) \) which satisfies equation (2).

Does this type of Bayesian Nash equilibrium exist for any set of parameter values? We can show by construction that the answer is no. For example, suppose that \( v \) and \( s \) are close to 1. Then the equality in equation (2) requires \( t_n \) to be close to \( q_n \). However, as \( s \) converges to 1 (i.e., as \( \sigma_n = (1-s)\pi_n \) converges to 0), \( t_n \) converges to 0 while \( q_n \) to a strictly positive value. Hence, \( t_n \) cannot be close to \( q_n \) enough to satisfy the equality for sufficiently large values of \( v \) and \( s \). The intuition is that if the invalid outcome is sufficiently attractive and if each voter is almost likely to prefer yes, then no-voters can hardly win the referendum and so strictly prefer abstaining. The similar argument also applies to \((y \rightarrow y&a, n \rightarrow n&a)\).

Finally, let us consider \((y \rightarrow y&a, n \rightarrow n&a)\). This strategy profile constitutes an equilibrium if there exists a pair of \( \pi_i \in (0,1) \ (i = y, n) \) which satisfies equation (1) with equality for \( i = y, n \). These two conditions are combined into

\[
\frac{p_y + t_y}{p_y + q_y} = \frac{p_n + t_n}{p_n + q_n} = v. \tag{3}
\]

As mentioned above, given a value of \( \pi_y \in (0,1) \), \( \pi_y \)'s convergence to 0 leads to \( y \)'s fraction converging to 1 while \( n \)'s fraction to 0. The opposite is also true: given a value of \( \pi_n \in (0,1) \), \( \pi_y \)'s convergence to 0 leads to \( y \)'s fraction converging to 0 while \( n \)'s fraction to 1. Therefore, for each set of parameter values, we can find a pair of \( \pi_i \in (0,1) \ (i = y, n) \) which satisfies the first equality in equation (3). Then the question is whether such a pair also satisfies the second equality for each value of \( v \). The answer is not necessarily.

Suppose that \( v \) is sufficiently small. Then the numerators of the two fractions \( p_i + t_i \ (i = y, n) \) must be sufficiently small: note that the denominators
satisfy \( p_y + q_y = p_n + q_n < 1 \) for any \( \pi_i \in (0,1) \) \((i = y,n)\) and parameter values. The convergence of the two numerators to 0 requires \( \pi_y \) and \( \pi_n \) to converge to 0. However, it is accompanied by the convergence of the denominators to 0. Hence, we need to know the limit of the two fractions. Suppose that we choose \( \pi_y \) and \( \pi_n \) which satisfy either \( \sigma_y > \sigma_n \) or \( \sigma_y < \sigma_n \). Then we must have \(|p_y - p_n| = |t_n - t_y| > 0\) so as to satisfy the first equality in equation (3). However, since \( t_i \) is of higher order than \( p_i \) and \( q_i \), this condition does not hold for sufficiently small values of \( \pi_y \) and \( \pi_n \). Hence, let \( \pi_y \) and \( \pi_n \) converge to 0 keeping \( \sigma_y = \sigma_n \). Then the first equality in equation (3) keeps holding for any values of \( \pi_y \), and for \( i = y,n \) we have in the limit

\[
\lim_{\pi_y \to 0, \sigma_y = \sigma_n} \frac{p_i + t_i}{p_i + q_i} = \frac{\sum_{k=0}^{[r_m]-1} \frac{1}{k!([r_m]-1-k)!}}{\sum_{k=0}^{[r_m]-1} \frac{1}{k!([r_m]-1-k)!}} = \frac{1}{2}.
\]

Therefore, the second equality in equation (3) does not hold for sufficiently small values of \( v \). The intuition is simple: if the invalid outcome is not attractive, members of at least one group must go to the poll with certainty. We have

**Proposition 2.** For \( r \in (1/m, (m-1)/m] \),

(i) \((y \to y,n \to n), (y \to a,n \to a), (y \to y,n \to a)\) and \((y \to a,n \to n)\) are Bayesian Nash equilibria for any parameter values;

(ii) \((y \to y,n \to n&a), (y \to y&a,n \to n)\) and \((y \to y&a,n \to n&a)\) are Bayesian Nash equilibria for a subset of parameter values;

(iii) \((y \to a,n \to n&a)\) and \((y \to y&a,n \to a)\) are never Bayesian Nash equilibria.

**3.2.2 Case \( r \in ((m-1)/m, 1] \)**

Finally, let us consider what happens if the lower bound goes up into the highest range where full voter turnout is required for the validity of the outcome (i.e., \([r_m] = m\)). Because even one voter can spoil the outcome by abstaining under \( r > (m-1)/m \), full voter turnout is harder to be realized: any voter whose favorite alternative is less likely to win should abstain. This incentive leads to the following proposition.

**Proposition 3.** For \( r > (m-1)/m \),

(i) \((y \to a,n \to a), (y \to y,n \to a)\) and \((y \to a,n \to n)\) are Bayesian Nash equilibria for any parameter values;
(ii) \((y \rightarrow y, n \rightarrow n), (y \rightarrow y, n \rightarrow n&\alpha), (y \rightarrow y&\alpha, n \rightarrow n)\) and \((y \rightarrow y&\alpha, n \rightarrow n&\alpha)\) are Bayesian Nash equilibria for a subset of parameter values;

(iii) \((y \rightarrow a, n \rightarrow n&\alpha)\) and \((y \rightarrow y&\alpha, n \rightarrow a)\) are never Bayesian Nash equilibria.

Proof: Strategy profiles \((y \rightarrow y, n \rightarrow n)\) and \((y \rightarrow y&\alpha, n \rightarrow n&\alpha)\) are proved below. See the Appendix for the other strategy profiles. Q.E.D.

There are two differences from the case of \(r \in (1/m, (m-1)/m]\). First, as suggested above, full voter turnout \((y \rightarrow y, n \rightarrow n)\) can happen only for a subset of parameter values under \(r > (m-1)/m\). Let us examine it. Because every vote is necessary for the validity of the outcome, we have \(t_y = t_n = 0\) for such \(r\)’s. Hence, the incentive constraint for full voter turnout is written as

\[
\max \left\{ \frac{q_y}{p_y}, \frac{q_n}{p_n} \right\} \leq \frac{1 - v}{v}.
\]

(4)

For what range of parameter values \(v\) and \(s\) is this condition easier to satisfy given \(m\) and \(r\)? Because the right-hand side converges to infinity as \(v\) converges to 0, inequality (4) holds for most values of \(s\) if \(v\) is sufficiently small. Small values of \(v\) mean small benefits from invalid outcomes, which induces voters to go to the poll.

The left-hand side is smaller as the values of \(q_y/p_y\) and \(q_n/p_n\) are closer to each other because \(q_y/p_y\) is decreasing in \(s\) while \(q_n/p_n\) is increasing in \(s\) (which comes from the fact that \(p_y\) and \(q_n\) are increasing in \(s\) while \(p_n\) and \(q_y\) are decreasing in \(s\)). So suppose \(q_y/p_y = q_n/p_n\), which holds at \(s = 1/2\). Then for \(i = y, n\) we have

\[
\frac{q_i}{p_i} = 1 - \frac{1}{\left(\frac{m-1}{2}\right)!} \sum_{k=0}^{m-1} \frac{1}{k!(m-1-k)!}.
\]

(5)

This formula is increasing in \(m\) and converges to 1 as \(m\) converges to infinity. So inequality (4) does not hold for sufficiently large values of \(v\). Even when \(m = 3\), it does not hold for \(v > 3/4\).

The second difference from the case of \(r \in (1/m, (m-1)/m]\) is that the strategy profile \((y \rightarrow y&\alpha, n \rightarrow n&\alpha)\) constitutes an equilibrium for a measure-zero set of parameter values. The incentive constraint for this strategy profile is

\[
\frac{q_y}{p_y} = \frac{q_n}{p_n} = \frac{1 - v}{v}.
\]
The first equality holds if and only if \( \sigma_y = \sigma_n \). Then equation (5) also holds here for any such pair of \( \pi_y \) and \( \pi_n \). Hence, only when \( m \) and \( v \) equate \( \frac{1}{v} \) with the right-hand side of equation (5), this strategy profile constitutes an equilibrium.

4 Numerical Example

In this section, we provide a numerical example. For simplicity of calculation, we set \( m = 13, r = 7/13 \) and \( s = 1/2 \). This is a case of \( r \in (1/m, (m-1)/m] \). We try to find a pair of \( \pi_y \) and \( \pi_n \) which realizes \( (y \rightarrow y&a, n \rightarrow n&a) \) in equilibrium for each value of \( v \). Let us focus on strategy profiles with \( \pi_y = \pi_n \) so that the first equality in equation (3) always holds.

<Figure 1 here>

Figure 1 describes the behavior of fraction \( \frac{p_i + t_i}{p_i + q_i} \mid_{\pi_n = \pi_y} \) as a function of \( \pi_y \). As we can see, the fraction is increasing in \( \pi_y \). Hence, equation (3) means that voters are more likely to go to the poll if the invalid outcome is more attractive. This might be counterintuitive, but it comes from the property of mixed-strategy equilibria. That is, if the invalid outcome is more attractive, going to the poll also must be more attractive so that voters use mixed strategies. In fact, larger voter turnout increases \( t_i \) relative to \( p_i \) and \( q_i \). The fraction converges to \( 21/32 \) as \( \pi_y \) converges to 0. Therefore, the mixed-strategy equilibrium with \( \pi_y = \pi_n \) exists only for \( v > 21/32 \).

5 The Ishikari Referendum

Ishikari (whose population was 56,000, respectively) is a city next to Sapporo (1,870,000) in Hokkaido Prefecture, Japan. Under the recent movement of municipal consolidation encouraged by the national government, Ishikari began discussion meetings in 2003 about whether to consolidate with two neighboring villages Atsuta (3,000) and Hamamasu (2,000); it had fourteen discussion meetings and more than ten explanation meetings to citizens. In the process, the mayor and most members of the city assembly showed their will to approve the consolidation. On the other hand, a questionnaire survey\(^5\) conducted by the city in September, 2004 revealed that the percentages

\(^{5}\)It is available in Japanese at http://www.city.ishikari.hokkaido.jp/kakubu/kzaisei/gappei/gappei-questionsaisyu.htm. In total 19,851 questionnaires were distributed to households in the city, and 5,939 (29.9\%) answered.
of households who answered approval, rather approval, rather disapproval, disapproval and others (i.e., not decided, indifferent or not interested) were 18.1%, 17.4%, 19.3%, 36.8% and 8.3% respectively.

A citizen group collected signatures from citizens and submitted to the mayor on September 27, 2004 a direct claim to have a referendum on the consolidation. The mayor presented a bill about the referendum to the assembly, and a bylaw was passed with modification; for the validity of the outcome, the voter turnout ratio must be greater than or equal to 60%. The imposition of lower bound was based on the opinions from councilmen that outcomes obtained under low voter turnout could not be said to reflect voters’ preferences (Hokkaido Shinbun, December 22, 2004). The value of 60% came from the voter turnouts in the previous elections of Ishikari (the mayor’s press conference, January 16, 2005).

The referendum was held on January 16, 2005. Of 44,879 voters, 19,450 actually voted; the voter turnout ratio was 43.34%. Therefore, the referendum became invalid and no ballot boxes were opened. According to Hokkaido Shinbun’s exit poll, 71 (35.5%) of 200 respondents voted for approval, 128 (64%) voted for disapproval, and 1 (0.5%) cast a blank vote (Hokkaido Shinbun, January 17, 2005).

Now we are ready to examine Ishikari voters’ incentives from our theoretical point of view. The concept of Bayesian Nash equilibrium itself does not predict which of multiple equilibria is realized. In the Ishikari referendum, however, we might be able to say that yes-voters had an incentive to abstain, which resulted in the low voter turnout, while no-voters to vote. These incentives were generated by the following two factors.

The first factor is voters’ expectation about the outcome of the referendum. The questionnaire survey showed that 35.5% of respondents approved or rather approved the consolidation while 56.1% disapproved or rather disapproved. This result created an expectation that disapproval would be the majority if the referendum was valid, which makes the strategy profile \((y \rightarrow a, n \rightarrow n)\) or \((y \rightarrow y&a, n \rightarrow n)\) more likely to realize; yes-voters try to spoil the outcome by abstaining while no-voters try to make the outcome valid by going to the poll.

The second factor is the benefit from the invalid outcome. Our model assumes for simplicity the symmetry of benefits from invalid outcomes be-

\footnote{It is available in Japanese at http://www.city.ishikari.hokkaido.jp/kaiken/H17/kaiken_h170116.htm.}

\footnote{The cost of going to the poll should be another important cause of abstention. In fact, the temperature on the day of the referendum was \(-5.2^\circ C\), which was 3.7\( ^\circ C\) lower than the average year (Hokkaido Shinbun, January 17, 2005). Such a cost must be taken into account if empirical research is conducted.}
between the two groups of voters; that is, the value $v \in (0, 1)$ is common for both groups. In the Ishikari referendum, however, the mayor and most councilmen revealed their stance of approval in advance of the referendum, which implied that they would proceed with the consolidation if the referendum was invalid. Hence, we can regard the value of $v$ as nearly 1 for yes-voters while nearly 0 for no-voters. Then the right-hand side of equation (1) is large for $i = y$, which induces yes-voters to abstain, while it is small for $i = n$, which induces no-voters to vote. Although this is a weak conjecture because of the small sample size and the different survey methods, the increase of the share of no-voters from 56.1% in the questionnaire to 64% at the exit poll might reflect such incentives.

6 Conclusion

We have constructed a Bayesian-Nash equilibrium model of voting decisions in an yes-no referendum whose outcome is valid only if the voter turnout is greater than a predetermined value. Our model shows that voters vote sincerely if the lower bound is not imposed or negligibly small, while strategic abstention can happen if it is effectively high. Therefore, the outcome generated with a sufficiently high lower bound may not reflect voters’ preferences accurately. Unless the full-voter-turnout equilibrium is highly expected, any other methods should be considered if election committees try to realize large voter turnouts.

We discussed our theoretical results in comparison with the Ishikari referendum. However, we have other referendums regarding municipal consolidation whose outcomes were invalid due to low voter turnout, and moreover imposing a lower bound on voter turnout is not limited to specific subjects but applied to any referendum in not a small number of cities and towns. Systematic empirical research on it is necessary.

The quorum in meetings has a similar property to the lower bound in referendums. That is, if the number of participants in a meeting is less than a predetermined number, they cannot have effective votes. In the Congress, for example, opposition party members sometimes do not appear so as to prevent bills proposed by the government party from being passed by vote. Our model can apply to this type of smaller voting, in which the coordination of behavior in each group is easier than in large elections. Such coordination might be described by introducing in each group a coordination device, or a leader as Shachar and Nalebuff (1999) do in their model of the U.S. presidential election. The allowance of coordination is expected to reduce the number of equilibria and yield a clearer prediction regarding the relationship between
the level of lower bound and the realization of voter turnout ratio. This extension is one future research task.

**Appendix**

**Proof of Proposition 3**: The same logic as the case of \( r \in (1/m, (m - 1)/m] \) implies that strategy profiles \((y \to a, n \to a), (y \to y, n \to a)\) and \((y \to a, n \to n)\) are Bayesian Nash equilibria while \((y \to a, n \to n\&a)\) and \((y \to y\&a, n \to a)\) are not.

Let us consider \((y \to y, n \to n\&a)\). This strategy profile constitutes an equilibrium if

\[
\frac{q_y}{p_y} \leq \frac{q_n}{p_n} = \frac{1 - v}{v}.
\]

Note that \(q_y/p_y\) is increasing in \(\pi_n\) while \(q_n/p_n\) decreasing. As \(\pi_n\) converges to 0, \(q_y/p_y\) converges to 0 while \(q_n/p_n\) to infinity. Hence, for sufficiently small values of \(v\), this incentive condition holds. If we try to make \(q_n/p_n\) as small as possible satisfying \(q_y/p_y \leq q_n/p_n\), we must have \(q_y/p_y = q_n/p_n\), which is attained by \(\pi_n = s/(1 - s)\). For such a value of \(\pi_n\), \(q_n/p_n\) equals the right-hand side of equation (5). Hence, this strategy profile is not equilibrium for sufficiently large values of \(v\). The similar logic applies to \((y \to y\&a, n \to n)\).

\[Q.E.D.\]

**References**


Figure 1. The Behavior of the LHS in Equation (3)